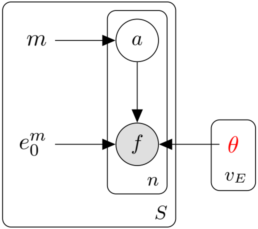
Dirichlet priors for IBM model 1

Wilker Aziz

April 11, 2019

MLE IBM 1

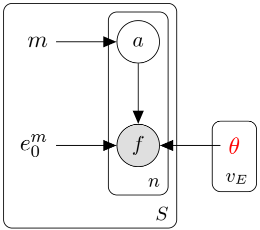


Global variables

- For each English type e,
  we have a vector θ_e of categorical parameters
  - $0 < \theta_e < 1$
  - $\sum_{f \in \mathcal{F}} \theta_{e,f} = 1$

  and $P_{F|E}(f|e) = \mathrm{Cat}(f|\theta_e) = \theta_{e,f}$

MLE IBM 1



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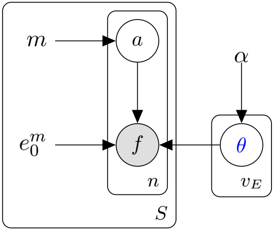
Local assignments

- For each French word position j,

$$A_j \sim \mathcal{U}(0 \dots m)$$

$$F_j | e_{a_j} \sim \mathrm{Cat}(\theta_{e_{a_j}})$$

Bayesian IBM 1

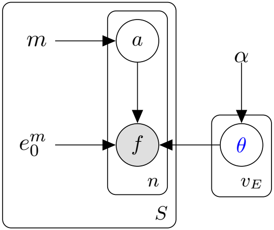


Global assignments

- For each English type e, sample categorical parameters

$$\theta_{\mathsf{e}} \sim \mathrm{Dir}(\alpha)$$

Bayesian IBM 1



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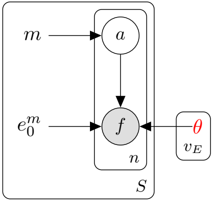

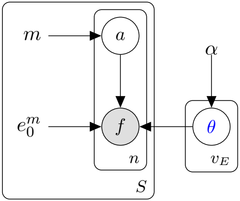

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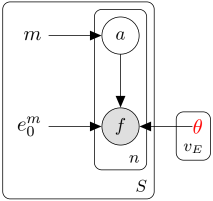
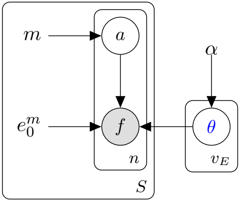
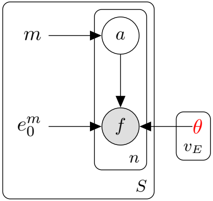


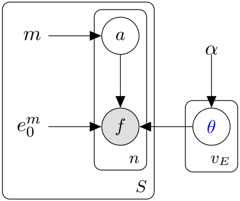


Incomplete data likelihood

$$P(f_1^n|e_1^m, \theta_1^{v_E}) = \prod_{j=1}^{n} \underbrace{\sum_{a_j=0}^{m} \overbrace{P(a_j|m)P(f_j|e_{a_j}, \theta_1^{v_E})}^{P(f_j, a_j|e_1^m, \theta_1^{v_E})}}_{P(f_j|e_1^m, \theta_1^{v_E})} \tag{1}$$

MLE vs Bayesian IBM1





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Marginal likelihood (evidence)

$$P(f_1^n|e_1^m, \alpha) = \int p(\theta_1^{v_E}|\alpha)P(f_1^n|e_1^m, \theta_1^{v_E})\mathrm{d}\theta_1^{v_E}$$

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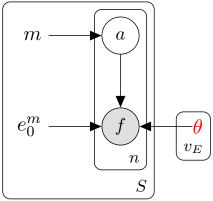

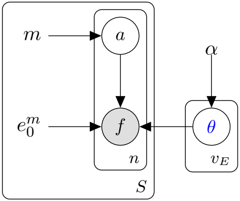

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What is a Dirichlet distribution?

Dirichlet: $\theta_e \sim \mathrm{Dir}(\alpha)$ with $\alpha \in \mathbb{R}^{v_F}_{>0}$

$$\mathrm{Dir}(\theta_e|\alpha) = \frac{\Gamma(\sum_{f\in\mathcal{F}} \alpha_f)}{\prod_{f\in\mathcal{F}} \Gamma(\alpha_f)} \prod_{f\in\mathcal{F}} \theta_{e,f}^{\alpha_f - 1} \tag{3}$$

- an exponential family distribution over probability vectors
- each outcome is a v_F-dimensional vector of probability values that sum to 1
- can be used as a prior over the parameters of a Categorical distribution
- that is, a Dirichlet sample can be used to specify a Categorical distribution
  e.g. $F|E = e \sim \mathrm{Cat}(\theta_e)$

Use this notebook and this wikipage to learn more

Why a Dirichlet prior on parameters?

If we set the components of α to the same value, we get a symmetric Dirichlet, if that value is small the Dirichlet will prefer

- samples that are very peaked
- in other words, categorical distributions that concentrate on few outcomes

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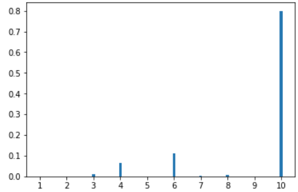
In Bayesian modelling we average over all possible parameters

- where each parameter set is weighted by a prior belief
- we can use this as an opportunity to, for example, express our preferences towards "peaked models"

Contrast the Dirichlet samples

```
plot_dirichlet_samples(alpha=0.1, nb_samples=1)
```

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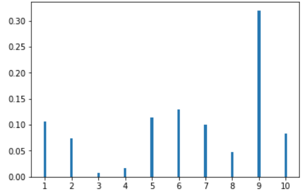

Top: sparse Dirichlet prior (small alpha)

- configurations that are this sparse will be roughly as likely
- less sparse configurations will be less likely
- "the prior doesn't care where the tall bars are, as long as they are few"

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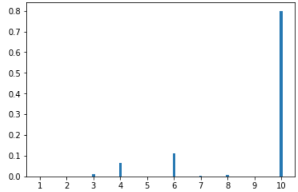


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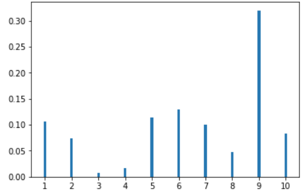

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Take samples from the top Dirichlet to parameterise a Categorical distribution conditioning on English word "dog"

- locations of the bars correspond to French words in the vocabulary
- the prior basically expresses the belief that whatever "dog" translates to, there shouldn't be many likely options available in French

An alternative way to write the likelihood

We can write a likelihood based on Categorical events as follows

$$
\begin{aligned}
P(f_1^n, a_1^n | e_1^m, \theta_1^{v_E}) &= \prod_{j=1}^{n} \underbrace{P(a_j|m)}_{\frac{1}{m+1}} \underbrace{P(f_j|e_{a_j}, \theta_1^{v_E})}_{\theta_{f_j|e_{a_j}}} \\
&= \frac{1}{(m+1)^n} \prod_{j=1}^{n} \theta_{f_j|e_{a_j}}
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I use $\theta_{e,f}$, $\theta_{e \to f}$, and $\theta_{f|e}$ interchangeably

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an alternative way iterates over the vocabulary of pairs, rather than over the sentence

$$P(f_1^n, a_1^n | e_1^m, \theta_1^{v_E}) \propto \prod_{\mathsf{e} \in \mathcal{E}} \prod_{\mathsf{f} \in \mathcal{F}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e} \to \mathsf{f} | f_1^n, a_1^n, e_1^m)} \tag{5}$$

where $\#(\mathsf{e} \to \mathsf{f} | f_1^n, a_1^n, e_1^m)$ counts how many times e and f are aligned in the sentence pair f_1^n, e_1^m given the alignments a_1^n

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An alternative way to write the likelihood (cont)

The new form reveals similarities to the Dirichlet

Dirichlet prior

$$p(\theta_1^{v_E}|\alpha) = \overbrace{\prod_{\mathsf{e}\in\mathcal{E}} \mathrm{Dir}(\theta_{\mathsf{e}}|\alpha)}^{\text{independent priors}} = \prod_{\mathsf{e}\in\mathcal{E}} \frac{\Gamma(\sum_{\mathsf{f}\in\mathcal{F}} \alpha_{\mathsf{f}})}{\prod_{\mathsf{f}\in\mathcal{F}} \Gamma(\alpha_{\mathsf{f}})} \prod_{\mathsf{f}\in\mathcal{F}} \theta_{\mathsf{f}|\mathsf{e}}^{\alpha_{\mathsf{f}}-1} \tag{6}$$

Multinomial (or Categorical likelihood)

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Thus

$$\begin{aligned} p(\theta_1^{v_E}, f_1^n, a_1^n|e_1^m, \alpha) &= p(\theta_1^{v_E}|\alpha) p(f_1^n, a_1^n|e_1^m, \theta_1^{v_E}) \\ &\propto \prod_{\mathsf{e}\in\mathcal{E}} \prod_{\mathsf{f}\in\mathcal{F}} \underbrace{\theta_{\mathsf{f}|\mathsf{e}}^{\alpha_{\mathsf{f}}-1} \times \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e}\to\mathsf{f}|f_1^n, a_1^n, e_1^m)}} \end{aligned}$$

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Bayesian IBM 1: Joint Distribution

Sentence pair: (e_0^m, f_1^n)

$$p(f_1^n, a_1^n, \theta_1^{v_E} | e_0^m, \alpha) = \overbrace{P(a_1^n|m)}^{\text{constant}} \underbrace{\prod_{\mathsf{e} \in \mathcal{E}}}_{\text{English types}} \overbrace{p(\theta_{\mathsf{e}}|\alpha)}^{\mathrm{Dir}\ \text{prior}} \overbrace{\prod_{\mathsf{e} \in \mathcal{E}} \underbrace{\prod_{\mathsf{f} \in \mathcal{F}}}_{\text{French types}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e} \to \mathsf{f} | f_1^n, a_1^n, e_1^m)}}^{\text{likelihood}}$$

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$$= P(a_1^n|m) \prod_{\mathsf{e}} \underbrace{\frac{\Gamma(\sum_{\mathsf{f}} \alpha_{\mathsf{f}})}{\prod_{\mathsf{f}} \Gamma(\alpha_{\mathsf{f}})} \prod_{\mathsf{f}} \theta_{\mathsf{f}|\mathsf{e}}^{\alpha_{\mathsf{f}}-1}}_{\text{Dirichlet}} \underbrace{\prod_{\mathsf{f}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e}\to\mathsf{f}|f_1^n, a_1^n, e_1^m)}}_{\text{Categorical}}$$

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$$\propto P(a_1^n|m) \prod_{\mathsf{e}} \prod_{\mathsf{f}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e} \to \mathsf{f} | a_1^n) + \alpha_{\mathsf{f}} - 1}$$

Bayesian IBM 1: Joint Distribution (II)

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Corpus: $(\mathbf{e}, \mathbf{f})$

$$\begin{aligned} p(\mathbf{f}, \mathbf{a}, \theta_1^{v_E} | \mathbf{e}, \mathbf{m}, \alpha) &\propto \prod_{(e_0^m, f_1^n, a_1^n)} P(a_1^n|m) \prod_{\mathsf{e}} \prod_{\mathsf{f}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e}\to\mathsf{f}|f_1^n, a_1^n, e_1^m) + \alpha_{\mathsf{f}} - 1} \\ &= P(\mathbf{a}|\mathbf{m}) \prod_{\mathsf{e}} \prod_{\mathsf{f}} \theta_{\mathsf{f}|\mathsf{e}}^{\#(\mathsf{e}\to\mathsf{f}|\mathbf{f}, \mathbf{a}, \mathbf{e}) + \alpha_{\mathsf{f}} - 1} \end{aligned} \tag{10}$$

where I use boldface to indicate the collection

Bayesian IBM 1: Inference

In Bayesian modelling **there is no optimisation**

- we **do not** pick one model
- instead, we infer a posterior distribution over unknowns and reason using all models (or a representative sample)

Bayesian IBM 1: Posterior

Intractable marginalisation

$$p(\mathbf{a}, \theta_1^{v_E} | \mathbf{e}, \mathbf{m}, \mathbf{f}, \alpha) = \frac{p(\mathbf{f}, \mathbf{a}, \theta | \mathbf{e}, \mathbf{m}, \alpha)}{\int \sum_{\mathbf{a}'} p(\mathbf{f}, \mathbf{a}', \theta' | \mathbf{e}, \mathbf{m}, \alpha) \mathrm{d}\theta'} \tag{11}$$

- $\theta_1^{v_E}$ are global variables: posterior depends on the entire corpus
- the summation goes over every possible alignment configuration for every possible parameter setting

Bayesian IBM 1: Approximate inference

Traditionally, we would approach posterior inference with an approximate algorithm such as Markov chain Monte Carlo

- based on sampling from the posterior by sampling one variable at a time and forming a chain whose stationary distribution is the true posterior

Mermer and Saraclar [2011] introduce Bayesian IBM1 and derive a Gibbs sampler

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MCMC is fully general, but can be hard to derive, and can be slow in practice

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Variational inference

Optimise an auxiliary model to perform inference

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\begin{aligned}
q* &= \underset{q \in \mathcal{Q}}{\arg\min} \; \mathrm{KL}(q(z) || p(z|x)) \\
&= \underset{q \in \mathcal{Q}}{\arg\min} \; \mathbb{E}_{q(z)}\left[\log \frac{q(z)}{p(z|x)}\right]
\end{aligned} \tag{12}
$$

Variational Inference - Objective

The original objective is intractable due to posterior

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\end{aligned}
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Variational Inference - Objective

The original objective is intractable due to posterior

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Evidence lowerbound (ELBO)

We've shown that minimising $\mathrm{KL}(q(z)||p(z|x))$ is equivalent to maximising a simpler objective

$$q* = \underset{q \in \mathcal{Q}}{\arg\max}\ \mathbb{E}_{q(z)}\left[\log p(z,x)\right] + \mathbb{H}(q(z))$$

known as the evidence lowerbound

The name ELBO has to do with the fact that $\log p(x) \geq$ ELBO

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For certain pairs of distributions in the exponential family, the quantities involved are both tractable

- e.g. the entropy of a Dirichlet variable is an analytical function of the parameter α
- e.g. check this lecture script for analytical results for the first term

The name ELBO has to do with the fact that $\log p(x) \geq$ ELBO

How do we design q for Bayesian IBM1?

Mean field assumption: make latent variables independent in q

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\begin{aligned}
q(a_1^n, \theta_1^{v_E}) &= q(\theta_1^{v_E}) \times Q(a_1^n) \\
&= \prod_{\mathsf{e}} q(\theta_{\mathsf{e}}) \times \prod_{j=1}^{n} Q(a_j)
\end{aligned}
\tag{13}
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Pick convenient parametric families

$$\begin{aligned} q(a_1^n, \theta_1^{v_E} | \phi, \lambda) &= \prod_{\mathsf{e}} q(\theta_{\mathsf{e}} | \lambda_{\mathsf{e}}) \times \prod_{j=1}^{n} Q(a_j | \phi_j) \\ &= \prod_{\mathsf{e}} \mathrm{Dir}(\theta_{\mathsf{e}} | \lambda_{\mathsf{e}}) \times \prod_{j=1}^{n} \mathrm{Cat}(a_j | \phi_j) \end{aligned} \tag{14}$$

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Find optimum parameters under the ELBO

- one Dirichlet parameter vector λ_{e} per English type
  λ_{e} consists of v_F strictly positive numbers
- one Categorical parameter vector ϕ_j per alignment link
  ϕ_j consists of a probability vector over $m + 1$ positions

ELBO for Bayesian IBM1

Objective

$$(\hat{\lambda}, \hat{\phi}) = \arg\max_{\lambda,\phi} \mathbb{E}_q[\log p(f_1^n, a_1^n, \theta_1^{v_E} | e_1^m, \alpha)] + \mathbb{H}(q)$$

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&= \arg\max_{\lambda,\phi} \sum_{j=1}^{m} \mathbb{E}_q[\log P(a_j|m) P(f_j|e_{a_j}, \theta_1^{v_E}) - \log Q(a_j|\phi_j)] \\
&\qquad + \sum_{\mathsf{e}} \underbrace{\mathbb{E}_q[\log p(\theta_e|\alpha) - \log q(\theta_e|\lambda_e)]}_{-\mathrm{KL}(q(\theta_e|\lambda_e)||p(\theta_e|\alpha))}
\end{aligned} \tag{15}
$$

VB for IBM1

Optimal $Q(a_j|\phi_j)$

$$\phi_{jk} = \frac{\exp\left(\Psi\left(\lambda_{f_j|e_k}\right) - \Psi\left(\sum_{\mathsf{f}} \lambda_{\mathsf{f}|e_k}\right)\right)}{\sum_{i=0}^{m} \exp\left(\Psi\left(\lambda_{f_j|e_i}\right) - \Psi\left(\sum_{\mathsf{f}} \lambda_{\mathsf{f}|e_i}\right)\right)} \tag{16}$$

where $\Psi(\cdot)$ is the digamma function

For derivations check Philip's notes

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Optimal $q(\theta_e|\lambda_e)$

$$\lambda_{\mathsf{f}|\mathsf{e}} = \alpha_{\mathsf{f}} + \sum_{(e_0^m, f_1^n)} \sum_{j=1}^{n} \mathbb{E}_{Q(a_j|\phi_j)}[\#(\mathsf{e} \rightarrow \mathsf{f}|f_j, a_j, e_1^m)] \tag{17}$$

For derivations check Philip's notes

Algorithmically

E-step as in MLE IBM1,
however, using $Q(a_j|\phi_j)$ instead of $P(a_j|e_0^m, f_j, \theta_1^{v_E})$

- maintain a table of parameters λ
- where in Frequentist EM you would use θ, use instead $\hat{\theta}$
- $\hat{\theta}_{\mathsf{f}|\mathsf{e}} = \exp\left(\Psi\left(\lambda_{\mathsf{f}|\mathsf{e}}\right) - \Psi\left(\sum_{\mathsf{f}'} \lambda_{\mathsf{f}'|\mathsf{e}}\right)\right)$
  (note these are not normalised probability vectors)

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M-step

- $\lambda_{\mathsf{f}|\mathsf{e}} = \alpha_{\mathsf{f}} + \mathbb{E}[\#(\mathsf{e} \to \mathsf{f})]$
  where expected counts come from E-step

References I

Coskun Mermer and Murat Saraclar. Bayesian word alignment for statistical machine translation. In *Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies*, pages 182–187, Portland, Oregon, USA, June 2011. Association for Computational Linguistics. URL http://www.aclweb.org/anthology/P11-2032.