Probabilistic Modelling

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Content

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    $P(\bigcup_i A_i) = \sum_i P(A_i)$
  - $P(\Omega) = 1$

Example

Consider the event of tossing a six-sided die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
We can define the simplest event space $F = \{\emptyset, \Omega\}$. Another event space is the set of all subsets of Ω.
For the first event space, the probability measure is given by $P(\emptyset) = 0$, $P(\Omega) = 1$.
For the second event space, one valid probability measure is to assign the probability of each set in the event space to be $\frac{i}{6}$ where i is the number of elements of that set; for example, $P(\{1, 2, 3, 4\}) = \frac{4}{6}$ and $P(\{1, 2, 3\}) = \frac{3}{6}$

Conditional probability

- Let B be an event with non-zero probability.
  The conditional probability of any event A given B is defined as:

$$P(A \mid B) = \frac{P(A, B)}{P(B)} \quad (1)$$

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- $P(A \mid B)$ is the probability measure of the event A after observing the occurrence of event B.

Chain rule

- Let $S_1, \cdots, S_k$ be events, $P(S_i) > 0$. Then the chain rule:

$$\begin{aligned} & P(S_1, S_2, \cdots, S_k) \\ = & P(S_1)P(S_2|S_1)P(S_3|S_2, S_1) \cdot P(S_k|S_1, S_2, \cdot S_{k-1}) \end{aligned} \tag{2}$$

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- In general, the chain rule is derived by applying the definition of conditional probability multiple times, for example:

$$\begin{aligned} & P(S_1, S_2, S_3, S_4) \\ =& P(S_1, S_2, S_3)P(S_4 \mid S_1, S_2, S_3) \\ =& P(S_1, S_2)P(S_3 \mid S_1, S_2)P(S_4 \mid S_1, S_2, S_3) \\ =& P(S_1)P(S_2 \mid S_1)P(S_3 \mid S_1, S_2)P(S_4 \mid S_1, S_2, S_3) \end{aligned} \tag{4}$$

Independence

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- Two events are called independent if and only if $P(A, B) = P(A)P(B)$, or $P(A \mid B) = P(A)$
- Thus, independence is equivalent to saying that observing B does not have any effect on the probability of A

Random variables

- We flip 10 coins, and we want to know the number of coins that come up heads.
  The sample space Ω are 10-length sequences of heads and tails. For example, we might have $\omega_0 = \langle H, H, T, H, T, H, H, T, T, T \rangle \in \Omega$.

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- we care about real-valued functions of outcomes, the number of heads that appear among our 10 tosses.
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- A random variable X is a function $X : \Omega \rightarrow \Re$.
- We will denote random variables using upper case letters X
- We will denote the value that a random variable may take on using lower case letters x.
  Thus, $X = x$ means that we are assigning the value $x \in \Re$ to the random variable X

Cumulative distribution functions

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- Properties:

$$\begin{aligned} 0 \leq F_X(x) \leq 1 \\ \lim_{x \rightarrow -\infty} F_X(x) = 0 \\ \lim_{x \rightarrow +\infty} F_X(x) = 1 \\ x \leq y \rightarrow F_X(x) \leq F_X(y) \end{aligned} \tag{6}$$

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- $p_X : \Omega \rightarrow \Re$ such that $p_X(x) = P(X = x)$
- Properties:

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\begin{aligned}
0 \leq p_X(x) \leq 1 \\
\sum_{x \in X} p_X(x) = 1 \\
\sum_{x \in A} p_X(x) = P(X \in A)
\end{aligned} \tag{7}
$$

Probability density functions

- For some continuous random variables, the cumulative distribution function $F_X(x)$ is differentiable everywhere.
  In these cases, we define the Probability Density Function or PDF as the derivative of the CDF

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{8}$$

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- Properties:

$$\begin{aligned} f_X(x) &\geq 0 \\ \int_{-\infty}^{\infty} f_X(x) &= 1 \\ \int_{x \in A} f_X(x)dx &= P(X \in A) \end{aligned} \tag{9}$$

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- If X is a continuous random variable with PDF $f_X(x)$, then the expected value of $g(X)$ is defined as:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{11}$$

Expectation

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- Properties:

$$\begin{aligned} &\mathbb{E}[a] = a \text{for any constant} a \in \Re \\ &\mathbb{E}[af(X)] = a\,\mathbb{E}[f(X)] \text{for any constant} a \in \Re \\ &\text{Linearity of Expectation}\, \mathbb{E}[f(X) + g(X)] = \mathbb{E}[f(X)] + \mathbb{E}[g(X)] \end{aligned} \tag{12}$$

Discrete random variables

- $X \sim$ Bernoulli(p) (where $0 \leq p \leq 1$):
  one if a coin with heads probability p comes up heads, zero otherwise

$$p(x) = \begin{cases} p, & \text{if } x = 1. \\ 1 - p, & \text{if } x = 0. \end{cases} \tag{13}$$

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- $X \sim \text{Binomial}(n, p)$ (where $0 \leq p \leq 1$):
  the number of heads in n independent flips of a coin with heads probability p

$$p = \binom{n}{x} \cdot p^x (1 - p)^{n-x} \tag{14}$$

Discrete random variables

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- $X \sim \text{Poisson}(\lambda)$ (where $\lambda > 0$):
  a probability distribution over the non-negative integers used for modelling the frequency of rare events.
  $$p(x) = e^{-\lambda}\frac{\lambda^x}{x!} \tag{16}$$

Continuous random variables

- $X \sim \text{Uniform}(a, b)$ (where $a < b$):
  equal probability density to every value between a and b on the real line

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- $X \sim \text{Normal}(\mu, \sigma^2)$: also known as the Gaussian distribution
  $$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{19}$$

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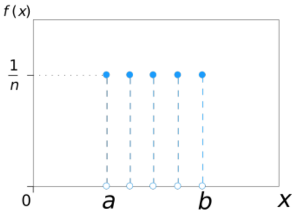
- We quantify the degree of belief we have in each outcome
- Uniform distribution: every outcome is equally likely
  if n is the size of the set of possible outcomes the probability that x takes on any value (e.g. a) is $\frac{1}{n}$

$$p(x) = \frac{1}{n}\text{for all} x \in [a, b] \quad (20)$$



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- Example: which pet do kids love the most?
  Sample space: $\Omega = \{\text{bird, cat, dog}\}$

$$x(\omega) = \begin{cases} 1 & \omega = bird \\ 2 & \omega = cat \\ 3 & \omega = dog \end{cases} \tag{21}$$

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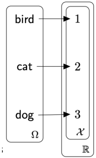
- we call $\mathcal{X}$ the support of X



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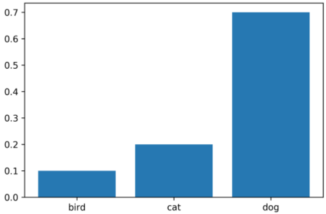
  $x \sim \text{Cat}(\theta_1, ..., \theta_k)$
- $x = 1, ..., k$
- the categorical parameter is a probability vector

$$0 \leq \theta_x \leq 1 \text{for} x \in [1, k]$$
$$\sum_{x=1}^{k} \theta_x = 1 \tag{22}$$



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  With the corresponding marginal distributions $p(x)$ and $p(y)$
- We denote the sum rule as (also known as the marginalization property):

$$p(x) = \begin{cases} \sum_{y \in Y} p(x, y), & \text{if} y \text{is discrete} \\ \int_{Y} p(x, y) dy, & \text{if} y \text{is continuous} \end{cases} \tag{23}$$

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- We sum out (or integrate out) the set of states y of the random variable Y.

Bayes' rule

- To derive expressions for conditional probability Bayes' rule

$$\underbrace{p(y \mid x)}_{\text{posterior}} = \frac{\overbrace{p(x \mid y)}^{\text{likelihood}} \, \overbrace{p(y)}^{prior}}{\underbrace{p(x)}_{evidence}} \tag{24}$$

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- If the random variables X and Y are continuous

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x \mid y)f(y)}{\int_{-\infty}^{\infty} f(x \mid y')f(y')dy'} \tag{26}$$

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  Given a probabilistic model, how do we obtain answers to relevant questions about the world?
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  Querying the marginal or conditional probabilities of certain events of interest.
- Learning
  Goal of fitting a model given a dataset. The model can be then use to make predictions about the future.

Bayesian networks

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- By the chain rule we can write any probability as:

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- A Bayesian network is a distribution in which each factor on the right hand side depends only on a small number of ancestor variables x_{A_i}:

$$p(x_i \mid x_{i-1}, ..., x_1) = p(x_i \mid x_{A_i}) \quad (28)$$

Bayesian networks

- Distributions of this form can be naturally expressed as directed acyclic graphs (DAG), in which vertices correspond to variables x_i and edges indicate dependency relationships.

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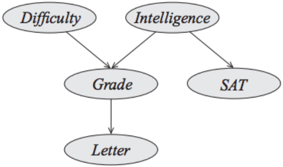
Model of a student's grade g on an exam. This grade depends on the exam's difficulty d and the student's intelligence i it also affects the quality l of the reference letter from the professor who taught the course. The student's intelligence i affects his SAT score s in addition to g. Each variable is binary, except for g, which takes 3 possible values.

Bayesian networks



$$p(l, g, i, d, s) = p(l \mid g)p(g \mid i, d)p(i)p(d)p(s \mid i) \tag{29}$$

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- probability p factorizes over a DAG G if it can be decomposed into a product of factors

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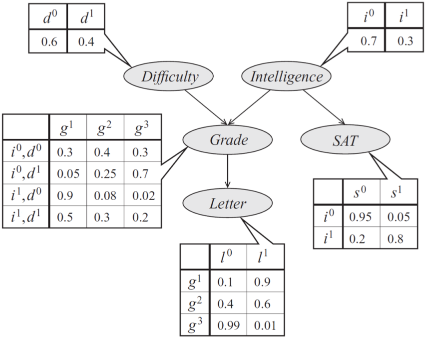

d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

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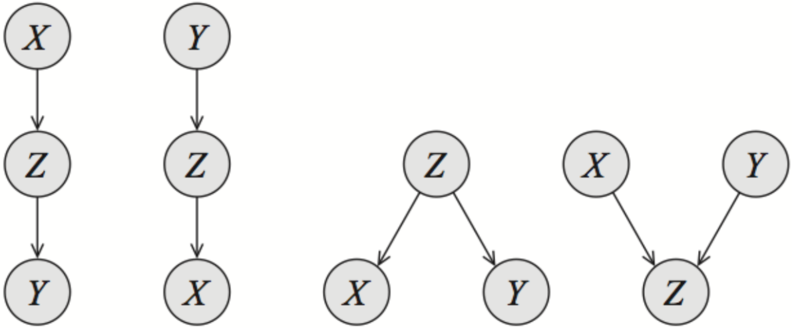

Probabilistic modelling

- Inference
  Given a probabilistic model, how do we obtain answers to relevant questions about the world?
  Querying the marginal or conditional probabilities of certain events of interest.

$$p(x_1) = \sum_{x_2} \sum_{x_3} ... \sum_{x_n} p(x_1, x_2, x3, ..., x_n) \tag{31}$$

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- Inference can be performed exactly.

Mixture models

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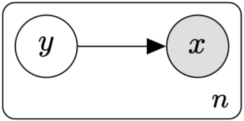
- We introduced the random variable y_i that ranges over the mixture components



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- We will learn the model from data, and use it to predict the existence of the missing word alignments

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- The generative story introduces the alignment variable a_j
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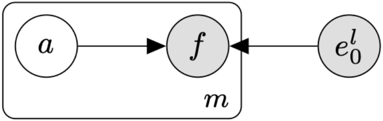
- The mixture components are English words

IBM graphical model



Questions?

References I